

Determining the Number of Holes of a 2D Digital Component is Easy

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Abstract. The number of holes in a connected component in 2D images is a basic invariant. In this note, a simple formula was proven using our previous results in digital topology [1,2]. The new is: $h = 1 + (|C_4| - |C_2|)/4$, where h is the number of holes, and C_i indicate the set of corner points having i direct adjacent points in the component.

Keywords: 2D digital space, Digital Gaussian curvature, Genus, Number of holes

1 Introduction

An image segmentation method can extract a connected component. A connected component S in a 2D digital image is often used to represent a real object. The identification of the object can be first done by determining how many holes in the component. For example, letter “A” has one hole and “B” has two holes. In other words, if S has h holes, then the complement of S has $h + 1$ connected components (if S does not reach the boundary of the image).

In this note, we provide two proofs for the following formula:

$$h = 1 + (|C_4| - |C_2|)/4$$

where h is the number of holes, and C_i indicate the set of corner points having i direct adjacent points in the component.

2 Some Concepts and Definitions of Digital Space

A digital space is a discrete space in which each point can be defined as an integer vector.

Two-dimensional digital space Σ_2 first. A point $P(x, y)$ in Σ_2 has two horizontal $(x, y \pm 1)$ and two vertical neighbors $(x \pm 1, y)$. These four neighbors are called directly adjacent points of p . p has also four diagonal neighbors: $(x \pm 1, y \pm 1)$. These eight (horizontal, vertical and diagonal) neighbors are called general (or indirect) adjacent points of p .

Let Σ_m be m -dimensional digital space. Two points $p = (x_1, x_2, \dots, x_m)$ and $q = (y_1, y_2, \dots, y_m)$ in Σ_m are directly adjacent points, or we say that p and q are direct neighbor if

$$d_D(p, q) = \sum_{i=1}^m |x_i - y_i| = 1.$$

p and q are indirectly adjacent points if

$$d_I(p, q) = \max_{1 \leq i \leq m} |x_i - y_i| = 1.$$

Note: “Indirectly adjacent points” include all directly adjacent points here. It may be the reason that we should change the word of “indirectly” to “generally.”

In a three-dimensional space Σ_3 , a point has six directly adjacent points and 26 indirectly adjacent points. Therefore, two directly adjacent points in Σ_3 are also called 6-connected, while two indirectly adjacent points are also called 26-connected. In this note, we mainly consider the direct adjacency. If we omit the word “direct,” “adjacency” means the direct adjacency.

A point in Σ_m is called a point-cell or 0-cell. A pair of points $\{p, q\}$ in Σ_m is called a line-cell or 1-cell, if p and q are adjacent points. A surface-cell is a set of 4 points which form a unit square parallel to coordinate planes. A 3-dimensional-cell (or 3-cell) is a unit cube which includes 8 points. By the same reasoning, we may define a k -cell. Fig. 2.1(a)(b)(c)(d) show a point-cell, line-cell, a surface-cell and a 3-cell, respectively.

Now let us consider to the concepts of adjacency and connectedness of (unit) cells. Two points p and q (point-cells, or 0-cells) are connected if there exists a simple path p_0, p_1, \dots, p_n , where $p_0 = p$ and $p_n = q$, and p_i and p_{i+1} are adjacent for $i = 1, \dots, n-1$.

Two cells are point-adjacent if they share a point. For example, line-cells $C1$ and $C2$ are point-adjacent in Fig. 2.1 (e), and surface-cells $s1$ and $s2$ are point-adjacent in Fig. 2.1(f). Two surface-cells are line-adjacent if they share a line-cell. For example, surface-cells $s1$ and $s3$ in Fig. 2.1(g) are line-adjacent.

Two line-cells are point-connected if they are two end elements of a line-cells path in which each pair of adjacent line-cells is point-adjacent. For example, line-cells $C1$ and $C3$ in Figure 2.1 (e) are point-connected. Two surface-cells are line-connected if they are two end elements of a surface-cells path in which each pair adjacent surface-cells are point-adjacent. For example, $s1$ and $s2$ in Fig. 2.1(f) are line-connected.

Two k -cells are k' -dimensional adjacent (k' -adjacent), $k > k' \geq 0$, if they share a k' -dimensional cell. A (simple) k -cells path with k' -adjacency is a sequence of k -cells v_0, v_1, \dots, v_n , where v_i and v_{i+1} are k' -adjacent and v_0, v_1, \dots, v_n are different elements. Two k -cells are called k' -dimensional connected if they are two end elements of a (simple) k -cells path with k' -adjacency.

Assume that S is a subset of Σ_m . Let $\Gamma^{(0)}(S)$ be the set of all points in S , and $\Gamma^{(1)}(S)$ be the line-cells set in S , ..., $\Gamma^{(k)}(S)$ be the set of k -cells of S . We say two elements p and q in $\Gamma^{(k)}(S)$ are k' -adjacent if $p \cap q \in \Gamma^{(k')}(S)$, $k' < k$.

Let $p \in \Sigma_3$, a line-neighborhood of p is a set containing p and its two adjacent points. A surface-neighborhood of p is a (sub-)surface where p is a inner point of the (sub-)surface.

Σ_m represents a special graph $\Sigma_m = (V, E)$. V contains all integer grid points in the m dimensional Euclidean space [?,1]. The edge set E of Σ_m is defined as $E = \{(a, b) | a, b \in V \& d(a, b) = 1\}$, where $d(a, b)$ is the distance between a and b . In fact, E contains all pairs of adjacent points. Because a is

an m -dimensional vector, $(a, b) \in E$ means that only one component, the i -th component, is different in a and b , $|x_i - y_i| = 1$, and the rest of the components are the same where $a = (x_1, \dots, x_m)$ and $b = (y_1, \dots, y_m)$. This is known as the direct adjacency. One can define indirect adjacency as $\max_i |x_i - y_i| = 1$. Σ_m is usually called an m -dimensional digital space. The basic discrete geometric element n -cells can be defined in such a space, such as 0-cells (point-cells), 1-cells (line-cells), and 2-cells (surface-cells).

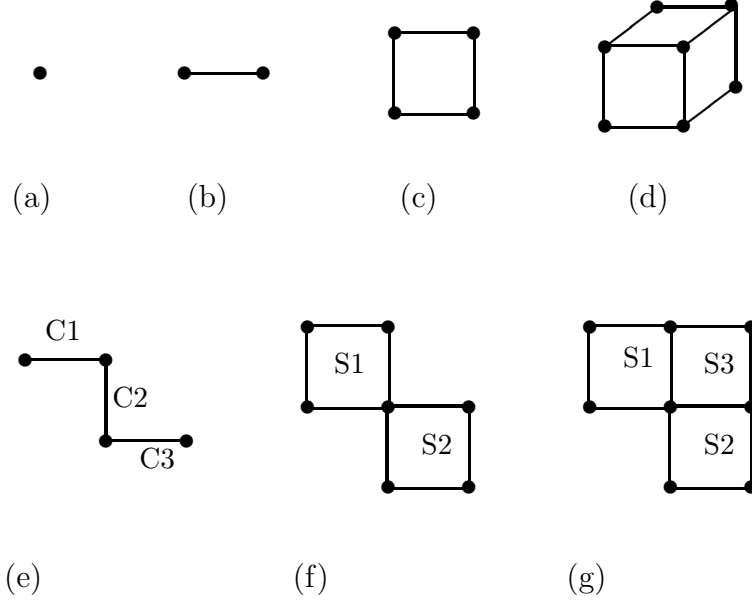


Fig. 1. Examples of basic unit cells and their connections : (a) 0-cells, (b) 1-cells, (c) 2-cells, (d) 3-cells, (e) point-connected 1-cells, (f) point-connected 2-cells, and (g) line-connected 2-cells.

3 Two Previous Related Results

We have proved some related theorem using Euler Characteristics and Gauss-Bonett Theorem. The first is about simple closed digital curves.

C is a simple closed curve where each element in C is a point in Σ_2 . In addition, C does not contain the following cases:

1 0
 0 1
 and
 0 1
 1 0

These two cases are called the pathological cases.

We use IN_C to represent the internal part of C . Since direct adjacency has the Jordan separation property, $\Sigma_2 - C$ will be disconnected.

We also call a point p on C a CP_i point if p has i adjacent points in $IN_C \cup C$. In fact, $|CP_1| = 0$ and $|CP_i| = 0$ if $i > 4$ in C .

CP_2 contains outward corner points, CP_3 contains straight-line points, and CP_4 contains inward corner points. For example, the following center point is a outward corner point:

```
0 0 0
0 1 1
0 1 x
```

But in next array, the center point is an inward corner point:

```
0 1 x
1 1 x
x x x
```

In [1], we showed for C ,

Lemma 1.

$$CP_2 = CP_4 + 4. \quad (1)$$

For a 3D image, Since cubical space with direct adjacency, or (6,26)-connectivity space, has the simplest topology in 3D digital spaces, we will use it as the 3D image domain. It is also believed to be sufficient for the topological property extraction of digital objects in 3D. In this space, two points are said to be adjacent in (6,26)-connectivity space if the Euclidean distance between these two points is 1.

Let M be a closed (orientable) digital surface in Σ_3 . in direct adjacency. We know that there are exactly 6-types of digital surface points [1][2].

Assume that M_i (M_3, M_4, M_5, M_6) is the set of digital points with i neighbors. We have the following result for a simply connected M [1]:

$$|M_3| = 8 + |M_5| + 2|M_6|. \quad (2)$$

We also have a genus formula based on the Gauss-Bonnet Theorem [2]

$$g = 1 + (|M_5| + 2 \cdot |M_6| - |M_3|)/8. \quad (3)$$

4 The Simple Formula for the Number of Holes in S

In this section, we first use the 3D formula to get the theorem for holes.

Let $S \subset \Sigma_2$ be a connected component and its boundary do not have the pathological cases. (We actually can detect those cases in linear time.)

We can embed S into Σ_3 to make a double S in Σ_3 . At $z = 1$ plane, we have a S , denoted S_1 , and we also have the exact same S at $z = 2$ plane, denoted S_2 .

Without loss generality, $S_1 \cup S_2$ is a solid object. (We here omit some technical details for the strict definition of digital surfaces.) It's boundary is closed digital surfaces with genus $g = h$. We know $g = 1 + (|M_5| + 2 \cdot |M_6| - |M_3|)/8$.

There will be no points in M_6 . We have

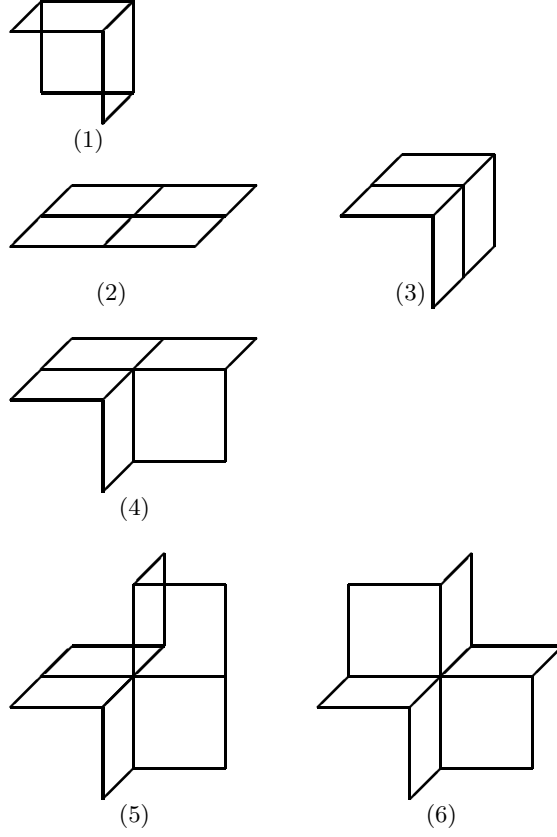


Fig. 2. Six types of digital surfaces points in 3D.

Theorem 1. *Let $S \subset \Sigma_2$ be a connected component and its boundary B is a collection of simple closed curves without pathological cases. Then, the number of holes in S is*

$$h = 1 + (C_4 - C_2)/4 \quad (4)$$

$C_4, C_2 \subset B$.

Proof: For each point x in C_2 in $C \subset S$ (C is the boundary of S), we will get two points in M_3 in $S_1 \cup S_2$. In the same way, if a point y is inward in $C_4 \in C$, we will get two points in M_5 in $S_1 \cup S_2$. There is no point in M_6 , i.e., $|M_6| = 0$. So $2|C_2| = |M_3|$, and $2|C_4| = |M_5|$. We have

$$h = g = 1 + (|M_5| + 2 \cdot |M_6| - |M_3|)/8 = 1 + (2|C_4| - 2|C_2|)/8$$

Thus,

$$h = 1 + (|C_4| - |C_2|)/4.$$

We can also prove this theorem using the curve theorem: $CP_2 = CP_4 + 4$ for a simple closed curve.

The Second Proof:

This can also be proved by the lemma in above section. $CP_2 = CP_4 + 4$;

A 2D connected component S with h holes that contains $h + 1$ simple closed curves in the boundary of S . Those curves do not cross each other.

The h curves corresponding to h holes will be considered oppositely in terms of inward-outward.

including one counts at inward and h is reversed outward with inward. It will get there.

Let $CP^{(0)}$ the outside curve of S and $CP^{(i)}, i = 1, \dots, h$, is the curve for the i -th hole.

Inward points to S is the outward points to $C^{(i)}, i = 1, \dots, h$. And vise versa.

$$CP_2^{(0)} = CP_4^{(0)} + 4$$

$$CP_2^{(i)} = CP_4^{(i)} + 4$$

The total outward points in the boundary of S is

$$CP_2 = CP_2^{(0)} + \sum_{i=1}^h CP_2^{(i)}.$$

The inward points in the boundary of S is

$$CP_4 = CP_4^{(0)} + \sum_{i=1}^h CP_4^{(i)}$$

Thus, $CP_4 - CP_2 = CP_4^{(0)} + \sum_{i=1}^h CP_4^{(i)} - CP_2^{(0)} - \sum_{i=1}^h CP_4^{(i)}$

$$\text{we have } CP_4 - CP_2 = -4 + \sum_{i=1}^h 4 = -4 + 4h$$

Therefore,

$$h = 1 + (CP_4 - CP_2)/4.$$

Therefore this formula is so simple to get the holes (genus) for a 2D object without any little sophisted algorithm, just count if the point is a corner point, inward or outward.

We could not get the similar simple formula in triangulated representation of the 2D object. This is the beauty of digital geometry and topology!

To test if this formula is correct, we can select the following examples

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (5)$$

In order to see clearly, we use “2” to represent points in CP_2 and use “4” to represent points in CP_4 .

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 2 & 0 & 0 \\ 0 & 2 & 4 & 4 & 1 & 2 & 0 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (6)$$

In this example $|CP_2| = 8$ and $|CP_4| = 4$. $h = 1 + (CP_4 - CP_2)/4 = 1 + (4 - 8)/4 = 0$.

Another example is the following

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (7)$$

In the second example $|CP_2| = 6$ and $|CP_4| = 6$. $h = 1 + (CP_4 - CP_2)/4 = 1 + (6 - 6)/4 = 1$.

When add a hole, we will add 4 more CP_4 points. That is the reason why this formula is correct.

5 Conclusion

In this paper, we have used digital topology to get a simple formula for calculating the number of holes in a connected component in 2D digital space. The formula is so simple and can be easily implemented. The author does not know if this formula was known or obtained already by other researchers.

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